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Geometric ‘tempus’ for a class of ergodic classical systems

Christian G Schroer
Institut für Festkörperforschung, Forschungszentrum Jülich, D-52425 Jülich, Germany

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Abstract. Using the symplectic formalism of classical mechanics, a smooth holonomy effect is identified in a special class of classical ergodic Hamiltonian systems (generalized canonical families) that are cycled adiabatically. It is a smooth shift in a time-resembling observable ‘tempus’ that is canonically conjugated to the phase space volume contained in the energy shell. The curvature 2-form of the adiabatic connection obtained is closely related to Hannay’s and the Robbins–Berry 2-form. The measurability of the ‘tempus’ is discussed.

1. Introduction

When he introduced the classical analogue of Berry’s phase for adiabatically cycled *integrable* Hamiltonian systems, Hannay [1] pointed out that it (‘Hannay’s angle’) had no straightforward generalization to systems with non-integrable (and possibly chaotic) dynamics. While Berry’s phase [2] and its generalizations [3–5] can be defined for all quantum systems that have bound states, Hannay’s angles exist only for those classical systems for which the motion at fixed external parameters is Liouville-integrable. Among all classical Hamiltonian systems these form a set of vanishing measure [6]. Since Hannay’s angles were shown to emerge from Berry’s phase when taking it to the classical limit [7], it is natural to search for a classical analogue of Berry’s phase for non-integrable (and possibly chaotic) systems. This search has not been successful, so far.

Using semiclassical methods Robbins and Berry [8] investigated systems with ergodic dynamics, taking Berry’s 2-form [2] to the semiclassical limit, but were not able to interpret their result in terms of holonomy [9]. From a classical point of view Hamiltonian systems that are ergodic on the energy shell are the best non-integrable systems to study holonomy, since a complete adiabatic theory exists for them [10]: the phase space volume that is enclosed by the energy shell is the only smooth adiabatic invariant. A holonomy effect in these systems can certainly not be a smooth shift on the energy shell, as pointed out by Golin *et al* [11]. The reason for this is the *sensitive dependence on initial conditions* that is inherent to ergodic systems with more than one degree of freedom.

In this paper a generalization of the so-called canonical families of ergodic Hamiltonian systems is considered, which were introduced by Robbins [9]. For canonical families there exists a parameter-dependent smooth canonical transformation, mapping the system at one parameter value to the system at another one. Even though this condition seems to be rather restrictive, many physically important situations are modelled by it. A rotated system, for example, is a canonical family, since a rotation in configuration space is canonical.

For these systems a time-resembling observable ‘tempus’ is constructed, which displays a smooth holonomy effect analogous to Hannay’s angles if the system is cycled adiabatically

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around a loop in parameter space. The result obtained is in full agreement with the above remark by Golin *et al* [11]. Even though the ergodic systems are as 'rare' as the integrable ones [6], and the generalized canonical families form an even smaller subset, they constitute the first type of chaotic systems for which holonomy can be defined.

In order to derive the equations of motion it is useful to embed the non-autonomous ergodic system into a larger autonomous system with two well separated time scales (*slow–fast system*), in which the slow configurations take the role of external parameters [12]. A suitable coordinate transformation of the fast subsystem for fixed slow configurations enables us to distinguish between dynamical and geometrical effects. The averaging of the phase space functions of interest requires their smoothness [10], which can be verified explicitly using this approach. Besides this technical advantage of the slow–fast systems, this description is also more general, since it includes the possibility of a reaction force of the fast subsystem onto the slow degrees of freedom, which is neglected in the non-autonomous case. This reaction force has been described by Berry and Robbins [13, 9], and an alternative derivation can be found in [14].

In the adiabatic limit the slow and the fast dynamics are separated and the slow degrees of freedom can be integrated by themselves. The trajectory of the slow configurations is introduced into the fast equations of motion.

The most important result of this paper is the existence of an observable of the ergodic fast degrees of freedom that displays a smooth holonomy effect when the slow configurations are cycled adiabatically. For integrable fast dynamics, holonomy was found to be a shift in the angle variables that are canonically conjugate to the adiabatically invariant actions [1]. Therefore for ergodic systems the quantity locally conjugate to the adiabatically invariant phase space volume is a good candidate to exhibit holonomy. For fixed external parameters this quantity is proportional to time and will therefore be called 'tempus' in what follows.

As for Hannay's angle, the geometric tempus cannot be defined for general curves in parameter space, since it is not independent of the choice of coordinates. If, however, the system is taken around a closed loop in parameter space, this gauge dependence is removed. In this case the 'tempus' divides naturally into a dynamical and a geometrical part, both of which are smooth and independent of the chaotic details of the motion. The deviations of the average from the real dynamical 'tempus' can in general hide the geometric effects, as was pointed out by Golin [15] for Hannay's angle. For generalized canonical families, however, these deviations are small and the geometric effects are indeed the first relevant correction to the dynamical tempus.

This paper is organized as follows. In section 2 ergodic Hamiltonian systems are discussed briefly, and the generalized canonical families are introduced. Section 3 describes Hamiltonian slow–fast systems with fast dynamics governed by a generalized canonical family. A transformation to coordinates is presented, in which the slow and fast variables separate in the adiabatic limit. Assuming the slow equations of motion to be solved, in section 4 we introduce the slow trajectory into the fast equations of motion, construct the 'tempus' and describe its dynamics. In addition to a dynamical part, we find a purely geometrical one that is analogous to Hannay's angle for the integrable case. To conclude, the measurability of the geometric tempus, its independence of the choice of coordinates and its geometrical interpretation are discussed.

2. Canonical families of ergodic Hamiltonian systems

Consider a Hamiltonian system with n degrees of freedom. Let the phase space $M \subset \mathbb{R}^{2n}$ be parametrized by canonical coordinates (p, q) and the Hamiltonian function be denoted by

$H(p, q) \in C^2(M)$. For suitable energies the motion is confined to a $(2n-1)$-dimensional energy shell $S(E) := \{(p, q) \in M | H(q, p) = E\}$. In the following we assume that an energy interval exists for which the motion is *ergodic* on almost any given energy shell, i.e. the time average of any smooth observable along almost any trajectory can be replaced by the microcanonical average of that observable. (Microcanonical averages will be denoted by $\langle \mathcal{A} \rangle = \int \mathrm{d}m_E \mathcal{A}(p, q)$, where $\mathrm{d}m_E$ denotes the microcanonical measure on the energy shell with energy E given by the initial conditions.) Instead of using the energy as a label, a given energy shell can as well be characterized by the phase space volume it contains,

$$\Omega(p, q) := \int \mathrm{d}p' \, \mathrm{d}q' \Theta(E(p, q) - H(p', q'))$$

provided that the motion is bounded and certain topological requirements are fulfilled [10] (Θ denotes the unit step function). We denote by $(\Omega_1, \Omega_2) \subset \mathbb{R}$ the interval of the phase space volume, for which the motion is ergodic on almost all energy shells labelled by $\Omega \in (\Omega_1, \Omega_2)$. The Hamiltonian $H(\Omega)$ is a strictly monotone smooth function of Ω only. (Note that a change of H as a function of Ω does not change the geometrical structure of the trajectories. Only the velocity at which the trajectory is followed by the system is changed.) In the following we will allow for an additional parameter dependence of the function $H(\Omega)$.

For later use, a new atlas $(U_j, (\kappa^{(j)}))$ ($j \in J$, J index set) is defined on the strip of energy shells $I := \{(p, q) \in M | (p, q) \in \bigcup_{\Omega \in (\Omega_1, \Omega_2)} S(H(\Omega))\}$, where the first coordinate κ_1 is identified with the phase space volume Ω: $\kappa_1 := \Omega$. The other coordinates are chosen such that changes from one chart to the other *do not* depend on κ_1, i.e. $(\partial \kappa_k^{(j)} / \partial \kappa_1)_{(i)} = 0$ ($k = 2, \ldots, 2n$, $j, i \in J$), and the microcanonical density ρ takes the form $\rho = \delta(\Omega_0 - \kappa_1)$, where Ω_0 is the phase space volume of the energy shell considered. (Note that these conditions do not determine the coordinates uniquely.)

We now introduce a slight generalization of the so-called canonical families [9]†. Consider a family of parameter-dependent smooth canonical transformations $\Phi(p, q, Q)$ of M onto itself, depending smoothly on k external parameters $Q \in \mathbb{R}^k$. The family of Hamiltonian systems generated by functions $H_2(p, q, Q) := H(\Omega(\Phi^{-1}(p, q, Q)), Q)$ is called a *generalized canonical family*. By construction the dynamics under H_2 are ergodic on the energy shells with $\Omega \in (\Omega_1, \Omega_2)$ for all Q, provided H is strictly monotone in Ω.

Besides Ω no other global invariants of the flow generated by $H(p, q)$ exist, and therefore the only symplectic flows Ψ on I leaving Ω invariant are generated by the Hamiltonian functions of type $\chi(\Omega, Q) \in C^2(I \times \mathbb{R}^k)$ [16], which generate a flow along the trajectories of the system (for example, $\Psi = \exp(-\{\chi, \bullet\})$, where $\{\bullet, \bullet\}$ denotes the Poisson bracket). Therefore the canonical transformations Φ are not uniquely determined for a given canonical family. In the construction of generalized canonical families we have the gauge freedom of choosing any $\Phi \circ \Psi$.

The reaction of the above system to adiabatic changes of the external parameters Q is described by Kazuga's adiabatic theorem for ergodic Hamiltonian systems [10], which can be outlined as follows. For almost all initial conditions on an appropriate energy shell the phase space volume Ω that is enclosed by the instantaneous energy shell is constant along the trajectory in the adiabatic limit (for details see [10]).

In the next section a generalized canonical family will be embedded into a slow–fast system by considering the parameters Q as configurations of k additional slow degrees of freedom.

† The notation has been changed compared to [9].

3. Hamiltonian slow–fast systems: adiabatic separation and fast equations of motion

Consider an autonomous Hamiltonian system that has k slow and n fast degrees of freedom, being parametrized by canonical coordinates (P, Q) and (p, q), respectively. The rather general Hamiltonian function

$$H(P, Q, p, q) := H_1(P, Q) + H_2(p, q, Q) \tag{1}$$

defines a Hamiltonian flow on the symplectic phase space by the canonical equations of motion (symplectic 2-form $\omega^2 = \mathrm{d}P \wedge \mathrm{d}Q + \mathrm{d}p \wedge \mathrm{d}q$).

In order to introduce the separation of time scales by a factor of the adiabatic slowness parameter ε explicitly, we choose the following symplectic 2-form ω^2:

$$\omega^2 := \frac{1}{\varepsilon}\,\mathrm{d}P \wedge \mathrm{d}Q + \mathrm{d}p \wedge \mathrm{d}q. \tag{2}$$

Then if H is of order zero in the adiabatic slowness parameter ε, the characteristic time scales for (P, Q) and (p, q) separate by ε, and $\varepsilon \to 0$ corresponds to the adiabatic limit. For $H_2(p, q, Q)$ we choose a generalized ergodic canonical family as given in the previous section.

We now separate the slow from the fast dynamics in the adiabatic limit. Besides the slow variables P and Q, the phase space volume Ω of the fast subsystem is also slow, i.e. its temporal change is of order ε. Therefore the system has effectively $(2k + 1)$ slow and $(2n - 1)$ fast variables, in which the equations of motion will be expressed in what follows. Using $\Phi^{-1}(p, q, Q)$ to pull back the set of energy shells $K_Q := \{(p, q) \in M|\Omega(\Phi^{-1}(p, q, Q)) \in (\Omega_1, \Omega_2)\}$ for any Q onto I defines an atlas $(U_j, (\kappa^{(j)}))$ on K_Q for every Q. By construction, the changes from one chart to the other do not depend on Q. The slow coordinates P and Q remain unchanged by the coordinate transformation.

In the new coordinates (P, Q, κ) the symplectic 2-form ω^2 and the Hamiltonian (1) are

$$\omega^2 = \frac{1}{\varepsilon}\,\mathrm{d}P_\alpha \wedge \mathrm{d}Q_\alpha + \sum_{i<j} B_{ij}\,\mathrm{d}\kappa_i \wedge \mathrm{d}\kappa_i + C_{i\alpha}\,\mathrm{d}Q_\alpha \wedge \mathrm{d}\kappa_i + \sum_{\alpha<\beta} F_{\alpha\beta}\,\mathrm{d}Q_\alpha \wedge \mathrm{d}Q_\beta \tag{3}$$

and

$$H(P, Q, \kappa_1) = H_1(P, Q) + H_2(\kappa_1, Q) \tag{4}$$

where the Lagrange brackets have been abbreviated as follows

$$\begin{aligned} B_{ij} &:= \frac{\partial p}{\partial \kappa_i}\frac{\partial q}{\partial \kappa_j} - \frac{\partial p}{\partial \kappa_j}\frac{\partial q}{\partial \kappa_i} \qquad C_{i\alpha} := \frac{\partial p}{\partial Q_\alpha}\frac{\partial q}{\partial \kappa_i} - \frac{\partial p}{\partial \kappa_i}\frac{\partial q}{\partial Q_\alpha} \\ F_{\alpha\beta} &:= \frac{\partial p}{\partial Q_\alpha}\frac{\partial q}{\partial Q_\beta} - \frac{\partial p}{\partial Q_\beta}\frac{\partial q}{\partial Q_\alpha} \end{aligned} \tag{5}$$

(here and in the following we use the summation convention: greek or roman indices run from 1 to k or 1 to $2n$, respectively). Note that, in general, only $C_{1\alpha}$ and $F_{\alpha\beta}$ are smooth functions on $K := \{(P, Q, p, q)|(p, q) \in K_Q\}$. The other $C_{i\alpha}$ and B_{ij} are not invariant under a change of chart. The Hamiltonian function (4) does not depend on the coordinates κ_i $(i = 2, \ldots, 2n)$.

Within the symplectic formalism [17] the following exact equations of motion for the slow–fast system are obtained (dotted quantities denote their derivative with respect to time)

$$\dot{P}_\alpha = -\varepsilon\frac{\partial H}{\partial Q_\alpha} + \varepsilon\left(B^{-1}\right)_{i1} C_{i\alpha}\frac{\partial H_2}{\partial \kappa_1} + \varepsilon^2\left(\underbrace{\left(B^{-1}\right)_{ij} C_{i\alpha} C_{j\beta}}_{\equiv -F_{\alpha\beta}} + F_{\alpha\beta}\right)\frac{\partial H_1}{\partial P_\beta} \tag{6a}$$

$$\dot{Q}_\alpha = \varepsilon \frac{\partial H_1}{\partial P_\alpha} \tag{6b}$$

$$\dot{\kappa}_i = \left(B^{-1}\right)_{i1} \frac{\partial H_2}{\partial \kappa_1} + \varepsilon \left(B^{-1}\right)_{ij} C_{j\alpha} \frac{\partial H_1}{\partial P_\alpha}. \tag{6c}$$

Note that the last term in (6*a*) vanishes exactly as can be shown by simple algebraic manipulation, and (B^{-1}) does not depend on Q for canonical families. The term $(B^{-1})_{i1}C_{i\alpha}$ is invariant under change of chart and is therefore a smooth function on phase space. Kazuga's adiabatic theorem therefore justifies averaging the above equations of motion for P, Q and κ_1, yielding an approximation for the slow motion to zeroth order in ε. Since $\langle (B^{-1})_{i1}C_{i\alpha}\rangle = 0$ (see [10]), κ_1 is constant to zeroth order and the only force acting from the fast subsystem onto the slow one is the Born–Oppenheimer potential force $\partial H_2/\partial Q_\alpha$.

Fixing a given set of coordinates (P, Q, κ), define

$$\nu^2 := C_{i\alpha}\ \mathrm{d}Q_\alpha \wedge \mathrm{d}\kappa_i + \sum_{\alpha<\beta} F_{\alpha\beta}\ \mathrm{d}Q_\alpha \wedge \mathrm{d}Q_\beta$$

as a 2-form over K that depends on the coordinates (P, Q, κ) chosen to define it. The 2-forms defined for different gauges transform into each other through

$$\nu^{2\prime} = \nu^2 - \frac{\partial}{\partial \kappa_1}\frac{\partial \chi}{\partial Q_\alpha}\,\mathrm{d}Q_\alpha \wedge \mathrm{d}\kappa_1$$

where $\chi(\kappa_1, Q)$ is the generating function of the gauge transformation as described in section 2. Since ν^2 is closed and K is assumed to be simply connected, ν^2 is exact on K such that a 1-form $A_\alpha\,\mathrm{d}Q_\alpha$ exists satisfying $\mathrm{d}A_\alpha \wedge \mathrm{d}Q_\alpha = \nu^2$. A change of gauge generated by $\chi(\kappa_1, Q)$ induces a change in A_α given by

$$A'_\alpha = A_\alpha + \frac{\partial}{\partial Q_\alpha}\chi(\kappa_1, Q). \tag{7}$$

Using a given A, define the scalar function $f(P, Q, \kappa) := -(1/\omega)\partial H/\partial P_\alpha(A_\alpha - \langle A_\alpha\rangle)$ on K ($\omega = \partial H_2/\partial\kappa_1$). Now perform a second coordinate transformation, choosing

$$\tilde{P}_\alpha := P_\alpha + \varepsilon A_\alpha \qquad \tilde{\kappa}_1 := \kappa_1 + \varepsilon f.$$

Expressed in these new coordinates the equations of motion of the slow variables become

$$\dot{\tilde{P}}_\alpha = -\varepsilon\frac{\partial H}{\partial Q_\alpha} + \varepsilon^2 \frac{\partial A_\beta}{\partial Q_\alpha}\frac{\partial H}{\partial P_\beta} \tag{8a}$$

$$\dot{Q}_\alpha = \quad \varepsilon\frac{\partial H}{\partial P_\alpha} \tag{8b}$$

$$\begin{aligned}\dot{\tilde{\kappa}}_1 = \quad & \varepsilon^2\left[\frac{\partial f}{\partial Q_\alpha}\frac{\partial H}{\partial P_\alpha} - \frac{\partial f}{\partial P_\alpha}\frac{\partial H}{\partial Q_\alpha}\right] + \varepsilon^2\left(B^{-1}\right)_{ij}\frac{\partial H}{\partial P_\alpha}\frac{\partial A_\alpha}{\partial \kappa_i}\frac{\partial f}{\partial \kappa_j} \\ & + \varepsilon^2\left(B^{-1}\right)_{1j}\frac{\partial H}{\partial \kappa_1}\frac{\partial A_\alpha}{\partial \kappa_j}\frac{\partial f}{\partial P_\alpha} + \mathrm{O}\left(\varepsilon^3\right).\end{aligned} \tag{8c}$$

The fast degrees of freedom ($i = 2, \ldots, 2n$) follow the equations of motion

$$\dot{\kappa}_i = \left(B^{-1}\right)_{i1}\omega - \varepsilon\left(B^{-1}\right)_{ij}\frac{\partial H}{\partial P_\alpha}\frac{\partial A_\alpha}{\partial \kappa_j} \tag{8d}$$

which will be needed in the following section. Averaging (8*a*), (8*b*) and (8*c*) yields the following effective Hamiltonian system for the slow degrees of freedom:

$$\omega^2 = \frac{1}{\varepsilon}\,\mathrm{d}\tilde{P}_\alpha \wedge \mathrm{d}Q_\alpha$$

$$H(\tilde{P} - \varepsilon\langle A\rangle(Q, \tilde{\kappa}_1), Q, \tilde{\kappa}_1) = H_1(\tilde{P} - \varepsilon\langle A\rangle(Q, \tilde{\kappa}_1), Q) + H_2(\tilde{\kappa}_1, Q)$$

for fixed adiabatic invariant $\tilde{\kappa}_1$. Since the higher-order terms in (8*c*) are bounded, they do not contribute in the order considered here. In first-order adiabatic approximation a gauge force generated by the gauge potential $\langle A\rangle$ known as 'geometric magnetism' acts on the slow freedoms in addition to the Born–Oppenheimer potential force, and the adiabatic invariant $\tilde{\kappa}_1$ is conserved to first order, since the terms of order ε^2 in (8*c*) vanish after averaging. This latter fact leads to the measurability of the first-order adiabatic correction to the tempus considered in the next section (see also [15]).

The slow dynamics to first order in the adiabatic slowness parameter have been investigated by Berry and Robbins in [18]. An alternative approach using the purely classical methods of this paper will published elsewhere [14].

4. Holonomy in ergodic dynamics: 'geometric tempus'

When deriving a geometric effect for ergodic fast dynamics two major obstacles must be overcome. Firstly, in contrast to holonomy for integrable systems, a geometric effect for the ergodic fast dynamics will not result in a smooth shift on the energy shell, because of sensitive dependence on initial conditions ($n > 1$) [11]. Even if the system is not changed parametrically the state of the system after an infinitely long ergodic evolution does not depend smoothly on the initial conditions. This is inherent to the chaotic evolution of ergodic systems with more than one degree of freedom. Therefore a search for a holonomic shift on the energy shell does not make sense. Furthermore, it would be tempting to investigate the coordinate canonically conjugate to the adiabatically invariant phase space volume $\kappa_1 = \Omega$, but for general ergodic systems ($n > 1$) such a global coordinate does not exist.

In spite of these complications, it is still possible to derive a holonomy in ergodic systems. For this purpose, we investigate the first differential ω^1 of a quantity θ that is locally canonically conjugate to the phase space volume κ_1 for *fixed* slow configurations Q. The local condition for canonical conjugation of θ and κ_1 is expressed in terms of the Poisson bracket of the fast subsystem

$$\delta_{1i} \overset{!}{=} \{\theta, \kappa_i\} = \omega^1\left(\left(B^{-1}\right) \mathrm{d}\kappa_i\right) \tag{9}$$

where the second equality follows from the symplectic formalism of Hamiltonian mechanics (see [17], p 215). Equation (9) determines the 1-form ω^1 uniquely

$$\omega^1 = B_{1i}\,\mathrm{d}\kappa_i \tag{10}$$

and B_{1i} is the smooth Lagrange bracket given by (5). Integrating ω^1 along the trajectory with some initial condition $(P^{(0)}, Q^{(0)}, \kappa^{(0)})$ and inserting (8*d*) for $\dot{\kappa}_i$ yields

$$\begin{aligned}\theta(t) &= \int \mathrm{d}t\, B_{1i}\dot{\kappa}_i = \int \mathrm{d}t\,\omega(\tilde{\kappa}_1, Q) - \varepsilon \int \mathrm{d}t\, \frac{\partial A_\alpha}{\partial \kappa_1}\frac{\partial H}{\partial P_\alpha} \\ &= \int \mathrm{d}t\,\omega(\tilde{\kappa}_1, Q) + \varepsilon \int \mathrm{d}t\, C_{1\alpha}\frac{\partial H}{\partial P_\alpha}\end{aligned} \tag{11}$$

where $(\omega(\tilde{\kappa}_1, Q) = \partial H_2/\partial\kappa_1)$ is an analogue of the angular velocity in integrable systems. When Q is fixed, the integral $\theta(t)$ is proportional to time and therefore the name 'tempus' is proposed for this quantity. (Note that the tempus is measured in units of $(\text{action})^{1-n}$ rather than in units of time.)

If Q is varied, the tempus depends not only on the initial conditions on the energy shell but also on the choice of κ-coordinates. In general its computation is rather complicated and depends on all the details of the motion. In the adiabatic limit an averaging theorem for smooth observables applies, and the computation of the tempus is greatly simplified. This averaging theorem can be outlined as follows.

In the adiabatic limit a time average of a smooth observable $\mathcal{A}$ along a generic trajectory can be replaced by the time average of the averaged observable along the trajectory of the averaged system:

$$\int \mathrm{d}t\,\mathcal{A}(P(t), Q(t), p(t), q(t)) \xrightarrow{\varepsilon\to 0} \int \mathrm{d}t\,\langle\mathcal{A}\rangle(\bar{P}(t), \bar{Q}(t)\kappa_1). \tag{12}$$

Here $\bar{P}(t)$ and $\bar{Q}(t)$ denote the trajectory of the averaged system. Since $C_{1\alpha}$ is a smooth function by construction of the atlas $(U_j, (\kappa)_j)$ in section 2, the tempus can be averaged. Averaging (11) yields

$$\bar{\theta} = \underbrace{\int \mathrm{d}t\,\omega(\tilde{\kappa}_1, Q(t))}_{=:\Delta\theta_{\mathrm{dyn}}} + \underbrace{\varepsilon\int_\gamma \mathrm{d}Q_\alpha\langle C_{1\alpha}\rangle_{\kappa_1, Q}}_{=:\Delta\theta_{\mathrm{geo}}} \tag{13}$$

and the tempus divides naturally into a *dynamical* ($\Delta\theta_{\mathrm{dyn}}$) and a *geometrical* part ($\Delta\theta_{\mathrm{geo}}$). γ denotes the curve in slow configuration space, along which Q evolves adiabatically. The geometric part does not depend explicitly on time but solely on the geometric properties of γ. Note that the error produced by averaging the dynamical part is smaller than first order, since $\tilde{\kappa}_1$ varies by terms of order ε^2. Therefore it is possible to resolve the geometric part, which is of order one in ε (see [15]).

For general curves γ the tempus and thus the geometric tempus cannot be defined gauge invariantly. Choosing closed curves (cycles) for γ, however, allows for defining a gauge invariant geometric tempus. Consider a gauge transformation generated by χ as described in section 2. $C_{1\alpha}$ transforms according to

$$C_{1\alpha} = C'_{1\alpha} + \frac{\partial}{\partial Q_\alpha}\frac{\partial\chi}{\partial\kappa_1} + \left(B^{-1}\right)_{i1} C_{i\alpha}\frac{\partial^2\chi}{\partial{\kappa_1}^2}$$

where the prime denotes the quantities calculated with respect to the new coordinates. The last term on the right vanishes (see p 3293) after averaging, and considering only cycles for γ makes the contribution from the closed form $(\partial/\partial Q_\alpha)/(\partial\chi/\partial\kappa_1)\mathrm{d}Q_\alpha$ vanish as well, provided the Q-space is simply connected. Thus the geometric tempus is invariant under gauge transformations described in section 2 for cycles γ.

A change of the atlas $(U_j, (\kappa)_j)$ is, in general, composed of two transformations: a local change of coordinates fulfiling $(\partial\kappa_k^{(j)}/\partial\kappa_1)_{(i)} = 0$ ($k = 2, \ldots, 2n,\ j, i \in J$) and a global smooth measure preserving transformation that leaves κ_1 unchanged. The first local transformation does not affect the quantities considered here. Under the second transformation we have

$$C_{1\alpha} = C'_{1\alpha} + \sum_{i=2}^{2n} C'_{i\alpha}\frac{\partial\tilde{\kappa}_i}{\partial\kappa_1} \tag{14}$$

where the primed quantities are calculated in the new coordinates. Averaging the last term in (14) and using (5) yields

$$\left\langle\sum_{i=2}^{2n} C'_{i\alpha}\frac{\partial\tilde{\kappa}_i}{\partial\kappa_1}\right\rangle = \frac{\partial}{\partial Q_\alpha}\left\langle\sum_{i=2}^{2n}\left(p\frac{\partial q}{\partial\tilde{\kappa}_i}\right)\frac{\partial\tilde{\kappa}_i}{\partial\kappa_1}\right\rangle - \left\langle\sum_{i=2}^{2n}\frac{\partial}{\partial\tilde{\kappa}_i}\left(p\frac{\partial q}{\partial Q_\alpha}\right)\frac{\partial\tilde{\kappa}_i}{\partial\kappa_1}\right\rangle.$$

The first term on the left is closed on Q-space and vanishes under integration around a closed loop γ. The other term can be transformed into

$$\left\langle \sum_{i=2}^{2n} \frac{\partial}{\partial\tilde{\kappa}_i}\left(p\frac{\partial q}{\partial Q_\alpha}\right)\frac{\partial\tilde{\kappa}_i}{\partial\kappa_1}\right\rangle = \left\langle \sum_{i=2}^{2n} \frac{\partial}{\partial\tilde{\kappa}_i}\left(p\frac{\partial q}{\partial Q_\alpha}\frac{\partial\tilde{\kappa}_i}{\partial\kappa_1}\right)\right\rangle - \left\langle \sum_{i=2}^{2n}\left(p\frac{\partial q}{\partial Q_\alpha}\right)\frac{\partial}{\partial\tilde{\kappa}_i}\frac{\partial\tilde{\kappa}_i}{\partial\kappa_1}\right\rangle \equiv 0.$$

Since the first term on the left is exact in $\tilde{\kappa}_i$ ($i = 2, \ldots, 2n$), its average vanishes because the energy shell has no boundary. The second term vanishes because T is measure preserving and thus $(\partial/\partial\tilde{\kappa}_i)/(\partial\tilde{\kappa}_i/\kappa_1) = 0$.

While the gauge transformations discussed in section 2 are local in Q, the latter transformation is global on Q-space. Note that, in contrast to the integrable case, there exists a global gauge dependence of $C_{1\alpha}$. The external derivative $\mathrm{d}\langle C_{i\alpha}\rangle$ $(\mathrm{d}\cdot = (\partial\cdot/\partial Q_\alpha)_\kappa\,\mathrm{d}Q_\alpha)$ is independent under both gauge transformations, and thus the geometric tempus is well defined for closed curves γ in Q-space.

Considering cycles γ only, the geometric tempus can be expressed as

$$\Delta\theta_{\mathrm{geo}} = -\frac{\partial}{\partial\kappa_1}\oint_\gamma \left\langle p\frac{\partial q}{\partial Q_\alpha}\right\rangle \mathrm{d}Q_\alpha \qquad (15)$$

and if the cycle γ can be contracted continuously to a point, the application of Stokes' theorem yields

$$\Delta\theta_{\mathrm{geo}} = -\frac{\partial}{\partial\kappa_1}\oint_\gamma \langle p\ \mathrm{d}q\rangle = -\frac{\partial}{\partial\kappa_1}\int_A \langle\,\mathrm{d}p \wedge \mathrm{d}q\rangle \qquad (16)$$

where A is any two-dimensional surface bordered by γ ($\delta A = \gamma$). Equation (16) is completely analogous to Hannay's 2-form. The main difference lies in the average, which is taken with respect to the microcanonical measure rather than over the ergodic measure on an n-torus, and the coordinates κ are held fixed when taking the external derivative with respect to Q rather than fixing the angle action variables (I, θ). The geometrical character of $\Delta\theta_{\mathrm{geo}}$ is clearly visible, because (16) does not depend on the dynamics of the system, but only on the geometrical properties of the energy shell. The 2-form obtained is identical with the Robbins–Berry 2-form for canonical families [8, 9].

In the case of one fast degree of freedom ($n = 1$) the phase space volume κ_1 coincides with 2π times the action I. κ_2 is proportional to some angle variable θ ($\kappa_2 = 1/(2\pi)\theta$). The tempus then coincides with $\theta/(2\pi)$ and the geometric tempus is $1/(2\pi)$ times Hannay's angle.

In the language of differential geometry the tempus can be interpreted as follows (see [11, 19]). The adiabatic transport defines a connection on the principal fibre bundle $G \times \mathbb{R}^k$ that has the group G of invariant flows Ψ as its fibre and the slow configuration space $\mathbb{R}^k$ as its base. In case of ergodic fast dynamics G is non-compact and isomorphic to $\mathbb{R}$ (see section 2 and [11]). The tempus is an element of the group G, and the geometric tempus (13) describes the adiabatic connection on the principal fibre bundle $G \times \mathbb{R}^k$, (16) being the holonomy of this connection.

This holonomy cannot be measured as a smooth shift in the final state of the system, since the final state depends sensitively on the initial conditions. The tempus must therefore be observed directly, requiring the observation of the fast dynamics during the whole evolution.

5. Conclusions

The symplectic formalism of Hamiltonian mechanics has been found to be a powerful tool to express the equations of motion in coordinates suited to separate slow and fast parts of the dynamics of slow–fast systems. The representation in these coordinates is crucial to distinguish between dynamical and geometrical effects.

For the first time a holonomy effect in a class of ergodic (and possibly chaotic) classical Hamiltonian systems has been derived, the 'geometric tempus'. It depends smoothly on initial conditions and is independent of the choice of coordinates on the energy shell. It is observable, but its measurement requires the observation of the system along the whole trajectory rather than the measurement of the initial and final state only, due to the sensitive dependence on initial conditions for chaotic systems.

One-dimensional ergodic systems are at the same time integrable and the geometric tempus differs from Hannay's angle only by a factor of $1/(2\pi)$. The 2-form describing the curvature of the adiabatic connection has a structure analogous to Hannay's 2-form [1, 7].

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